A Liouville theorem for solutions of degenerate Monge-Ampère equations

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Abstract

In this paper, we give a new proof of a celebrated theorem of Jörgens which states that every classical convex solution of

$$\det \nabla^2 u(x) = 1 \quad \text{in } \mathbb{R}^2$$

has to be a second order polynomial. Our arguments do not use complex analysis, and can be applied to establish such Liouville type theorems for solutions of a class of degenerate Monge-Ampère equations. We prove that every convex generalized (or Alexandrov) solution of

$$\det \nabla^2 u(x_1, x_2) = |x_1|^\alpha \quad \text{in } \mathbb{R}^2,$$

where $\alpha > -1$, has to be

$$u(x_1, x_2) = \frac{a}{(\alpha+2)(\alpha+1)}|x_1|^{2+\alpha} + \frac{ab^2}{2}x_1^2 + bx_1x_2 + \frac{1}{2a}x_2^2 + \ell(x_1, x_2)$$

for some constants $a > 0$, b and a linear function $\ell(x_1, x_2)$.

This work is motivated by the Weyl problem with nonnegative Gauss curvature.

1 Introduction

A celebrated theorem of Jörgens states that every entire classical convex solution of

$$\det \nabla^2 u(x) = 1 \tag{1}$$

in $\mathbb{R}^2$ has to be a second order polynomial. This theorem was first proved by Jörgens [20] using complex analysis methods. An elementary and simpler proof, which also uses complex analysis, was later given by Nitsche [23], where Bernstein theorem for two dimensional minimal surfaces is established as a corollary. Jörgens' theorem was extended to smooth convex solutions in higher dimensions by Calabi [8] for dimension ≤ 5 and by Pogorelov [26] for all dimensions.

Another proof was given by Cheng and Yau [9] along the lines of affine geometry. Note that each local generalized solution of (1) in dimension two is smooth, but this is false in dimension ≥ 3. Caffarelli [4] established the Jörgens-Calabi-Pogorelov theorem for generalized solutions (or viscosity solutions). Trudinger-Wang [27] proved that the only convex open subset Ω of $\mathbb{R}^n$ which admits a convex C^2 solution of (1) in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$ is $\Omega = \mathbb{R}^n$. Caffarelli-Li [6] established the asymptotical behaviors of viscosity solutions of (1) outside of a bounded convex subset of $\mathbb{R}^n$ for $n \geq 2$ (the case $n = 2$ was studied before in Ferrer-Martínez-Milán [12, 13] using complex analysis), from which the Jörgens-Calabi-Pogorelov theorem follows.

In this paper, we provide a new proof of this Jörgens' theorem. Our arguments do not use complex analysis. This allows us to establish such Liouville type theorems for solutions of a class of degenerate Monge-Ampère equations. More precisely, we classify entire convex solutions of the degenerate Monge-Ampère equations

$$\det \nabla^2 u(x_1, x_2) = |x_1|^\alpha \quad \text{in } \mathbb{R}^2, \tag{2}$$

where $\alpha > -1$. The equation (2) appears, for instance, as a blowup limiting equation of

$$\det \nabla^2 u(x_1, x_2) = (x_1^2 + x_2^2)^{\alpha/2} \tag{3}$$

in Daskalopoulos-Savin [10] in the study of the Weyl problem with nonnegative Gauss curvature.

In 1916, Weyl [28] posed the following problem: Given a Riemannian metric g on the 2-dimensional sphere $\mathbb{S}^2$ whose Gauss curvature is positive everywhere, does there exist a global C^2 isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, \mathrm{d}s^2)$, where $\mathrm{d}s^2$ is the standard flat metric on $\mathbb{R}^3$?

Lewy [21] solved the problem in the case that g is real analytic. In 1953, Nirenberg [22] gave a solution to this problem under the regularity assumption that g has continuous fourth order derivatives. The result was later extended to the case that g has continuous third order derivatives by Heinz [17]. An entirely different approach was taken independently by Alexandrov and Pogorelov; see [1, 24, 25].

There are also work (see [19, 14, 18, 10]) which study the problem with nonnegative Gauss curvature. Guan-Li [14] showed that for any C^4 metric on $\mathbb{S}^2$ with nonnegative Gauss curvature, there always exists a global $C^{1,1}$ isometric embedding into $(\mathbb{R}^3, \mathrm{d}s^2)$; see also Hong-Zuily [18] for a different approach to this $C^{1,1}$ embedding result. Guan and Li asked there that whether the $C^{1,1}$ isometric embeddings can be improved to be $C^{2,\gamma}$ or even $C^{2,1}$. The problem can be reduced to regularity properties of solutions of a Monge-Ampère equation that becomes degenerate at the points where the Gauss curvature vanishes. If the Gauss curvature of g only has one nondegenerate zero, the regularity of the isometric embedding amounts to studying the regularity of solutions of (3) near the origin for $\alpha = 2$, and it has been proved in Daskalopoulos-Savin [10] that the solutions of (3) are $C^{2,\gamma}$ near the origin for $\alpha > 0$.

A comprehensive introduction to the Weyl problem and related ones can be found in the monograph Han-Hong [16].

The main result of this paper is the following:

Theorem 1.1. *Let u be a convex generalized (or Alexandrov) solution of* (2) *with* $\alpha > -1$. *Then there exist some constants* $a > 0$, b *and a linear function* $\ell(x_1, x_2)$ *such that*

$$u(x_1, x_2) = \frac{a}{(\alpha+2)(\alpha+1)}|x_1|^{2+\alpha} + \frac{ab^2}{2}x_1^2 + bx_1x_2 + \frac{1}{2a}x_2^2 + \ell(x_1, x_2).$$

Recall that every generalized solution of (1) in an open subset of $\mathbb{R}^2$ is strictly convex (and thus, smooth). However, this is not the case for generalized (or even classical) solutions of $\det \nabla^2 u = |x_1|^\alpha$ when $\alpha > 0$; see Example 4.3. And it follows from [3] that the generalized solutions of such equations with homogenous boundary condition are strictly convex.

The paper is organized as follows. To illustrate our method, in Section 2 we first present another proof of Jörgens' theorem, which only makes use of a few properties of harmonic functions. Those properties also hold in general for solutions of elliptic or even certain degenerate elliptic equations, such as a Grushin type equation shown in Section 3 that the partial Legendre transform of u satisfies. In Section 4, we show that entire solutions of (2) are strictly convex and prove Theorem 1.1.

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2 A new proof of Jörgens' theorem

Proof of Theorem 1.1 when $\alpha = 0$. First of all, we know that u is smooth. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x_1, x_2) = (x_1, \nabla_{x_2} u(x)) =: (p_1, p_2). \tag{4}$$

Clearly, T is injective. Recall that the partial Legendre transform $u^*(p)$ is defined as

$$u^*(p) = x_2 \nabla_{x_2} u(x) - u(x).$$

Then

- u^* is concave w.r.t. p_1 and convex w.r.t. p_2;
- $(u^*)^* = u$;
- $\Delta u^* = 0$ in $T(\mathbb{R}^2)$.

Step 1: Prove the theorem under the assumption $T(\mathbb{R}^2) = \mathbb{R}^2$.

For simplicity, we will denote $\nabla_{x_i} u(x), \nabla_{p_i} u^*(p)$ as $u_i(x), u_i^*(p)$ respectively throughout the paper if there is no possibility of confusion. Since u^* is convex w.r.t. p_2, we have

$$u_{22}^* \geq 0 \quad \text{and} \quad \Delta u_{22}^* = 0 \quad \text{in} \quad \mathbb{R}^2.$$

It follows from Liouville theorem for entire nonnegative harmonic functions that $u^*_{22} = a \geq 0$ for some constant a. By the equation of u^*, we have $u^*_{11} = -a$. Hence,

$$u^* = (-p_1^2 + p_2^2)a/2 + bp_1p_2 + \ell(p_1, p_2)$$

for some constant b and linear function ℓ. Since $u = (u^*)^*$, $a > 0$ and we are done.

Step 2: Prove $T(\mathbb{R}^2) = \mathbb{R}^2$.

We prove it by contradiction. Suppose that there exists $\bar{x}_1$ such that

$$\lim_{x_2 \to +\infty} u_2(\bar{x}_1, x_2) := \beta < +\infty.$$

Claim: for any $x_1 \in \mathbb{R}$,

$$\lim_{x_2 \to +\infty} u_2(x_1, x_2) = \beta.$$

Indeed, by the convexity of u, for $t > 0$

$$u(\bar{x}_1, 0) + t\beta \geq u(\bar{x}_1, t) \geq u(x_1, x_2) + u_1(x)(\bar{x}_1 - x_1) + u_2(x)(t - x_2),$$

namely,

$$u_2(x)(1 - x_2/t) \leq \beta + \frac{1}{t}\{u(\bar{x}_1, 0) - u(x_1, x_2) - u_1(x)(\bar{x}_1 - x_1)\}.$$

Sending $t \to \infty$, we have $u_2(x_1, x_2) \leq \beta$. Hence, $\lim_{x_2 \to +\infty} u_2(x_1, x_2) \leq \beta$. Repeating this argument with x_1 and $\bar{x}_1$ exchanged, we would see that $\lim_{x_2 \to +\infty} u_2(x_1, x_2) \geq \beta$.

Without loss of generality, we assume that $\beta = 1$. Therefore,

$$T(\mathbb{R}^2) = (-\infty, +\infty) \times (\beta_0, 1)$$

for some $-\infty \leq \beta_0 < 1$. Since T is one-to-one and $u^*_2(p_1, p_2) = x_2$, we have

$$\lim_{p_2 \to 1^-} u^*_2(p_1, p_2) = +\infty,$$

i.e., for any $C > 2$, there exists ε (may depend on $\bar{p}_1$ which is arbitrarily fixed) such that $u^*_2(\bar{p}_1, p_2) \geq C$ for every $p_2 \geq 1 - \varepsilon$. By continuity of u^*_2, $u^*_2(p_1, 1 - \varepsilon) \geq C - 1$ for $p_1 \in (\bar{p}_1 - \delta, \bar{p}_1 + \delta)$ for some small δ. Since u^*_2 is monotone increasing in p_2, we have $u^*_2(p_1, p_2) \geq C - 1$ in $(\bar{p}_1 - \delta, \bar{p}_1 + \delta) \times (1 - \varepsilon, 1)$. This shows that

$$\lim_{(p_1, p_2) \to (\bar{p}_1, 1)} u^*_2(p_1, p_2) = +\infty$$

for any $\bar{p}_1 \in \mathbb{R}$, and in particular, u^*_2 is positive near the point $(2, 1)$. Without loss of generality, we may assume that u^*_2 is positive in $[1, 3] \times [0, 1)$. For any $C > 0$ large, we let

$$v(p_1, p_2) := u^*_2(p_1, p_2) - Cp_2(p_1 - 1)(3 - p_1) - \frac{C}{3}p_2^3 + \frac{C}{3}.$$

Since $\Delta u_2^* = 0$, it follows that $\Delta v = 0$. By the maximum principle, $v \geq 0$ in $[1,3] \times [0,1)$. In particular, $v(2, \bar{p}_2) \geq 0$ where $\bar{p}_2 \in (0,1)$ is chosen such that

$$\bar{p}_2 + \bar{p}_2^3/3 - 1/3 = 1/2.$$

Hence, $u_2^*(2, \bar{p}_2) \geq C/2$ for all $C > 0$, which is a contradiction. $\square$

3 Homogenous Grushin type equations

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ such that $\Omega \cap \{x | x_1 = 0\} \neq \emptyset$. Consider

$$Lu := u_{x_1x_1} + |x_1|^\alpha u_{x_2x_2} = 0 \quad \text{in } \Omega, \tag{5}$$

where $\alpha > -1$. We will see later that the partial Legendre transform of solutions of (2) satisfies (5). Also, (5) appears in [7] in extension formulations for fractional Laplacian operators.

Definition 3.1. *We say a function u is a strong solution of* (5) *if* $u \in C^1(\Omega) \cap C^2(\Omega \setminus \{x_1 = 0\})$ *and satisfies*

$$Lu = 0 \quad \textit{in } \Omega \setminus \{x_1 = 0\}.$$

In this following, we will see that our definition of strong solution coincides with the classical strong solutions. Indeed, $u \in W^{2,p}_{loc}$ for any $1 \leq p < -\frac{1}{\alpha}$ if $\alpha \in (-1, 0)$, and u is $C^{2,\delta}$ if $\alpha \geq 0$. We have to be careful if we want to study continuous viscosity solutions of (5) which may not have uniqueness property, see Remark 4.3 in [7]. However, L^p-viscosity solutions of certain elliptic equations with coefficients deteriorating along some lower dimensional manifolds would be such strong solutions, see, e.g., [29]. The following proposition is in the same spirit of Lemma 4.2 in [7]. For regularity properties of solutions of a more general class of quasilinear degenerate elliptic equations we refer to [11].

Proposition 3.2. *For any $g \in C(\partial\Omega)$, there exists a unique strong solution u of* (5) *with $u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Furthermore, we have*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} g, \quad \min_{\overline{\Omega}} u \geq \min_{\partial\Omega} g, \tag{6}$$

and, for any $\Omega' \subset\subset \Omega$ and $k \in \mathbb{N}$,

$$\sum_{l=0}^{k} \|\nabla_{x_2}^l u\|_{C^1(\Omega')} \leq C\|g\|_{C^0(\partial\Omega)}, \tag{7}$$

where $C > 0$ depends only on $n, \alpha, k, dist(\Omega', \partial\Omega)$.

Proof. Uniqueness. Clearly, the uniqueness would follow from (6). The proof of uniqueness in Lemma 4.2 in [7] can be applied to obtain (6) and we include it for completeness. Let u be a strong solution of (5) with $u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Let $v = u - \max_{\partial\Omega} g + \varepsilon|x_1|$, where ε is small. Suppose v has an interior maximum point $\bar{x}$ in Ω. Then $\bar{x}_1 = 0$, since otherwise v satisfies an elliptic equation near $\bar{x}$ which does not allow an interior maximum point. On the other hand, if $\bar{x}_1 = 0$, then $\bar{x}$ can not be a maximum point of v since $\partial_+ v(\bar{x}) > \partial_- v(\bar{x})$. Therefore, the maximum of v is achieved on $\partial\Omega$, i.e. $u - \max_{\partial\Omega} g + \varepsilon|x_1| \leq \varepsilon\, \text{diam}(\Omega)$. Sending $\varepsilon \to 0$, we obtain $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} g$. Similarly, we can show that $\min_{\overline{\Omega}} u \geq \min_{\partial\Omega} g$.

Existence. For $\varepsilon > 0$ sufficiently small, let $0 < \eta_\varepsilon(x_1) \in C^\infty(-\infty, \infty)$ such that

$$\eta_\varepsilon(x_1) = |x_1|^\alpha \quad \text{for} \quad |x_1| > 2\varepsilon; \quad \eta_\varepsilon(x_1) = \varepsilon^\alpha \quad \text{for} \quad |x_1| \leq \varepsilon.$$

By the standard linear elliptic equation theory, there exists a unique solution $u^\varepsilon \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ of

$$L_\varepsilon u^\varepsilon := u^\varepsilon_{x_1x_1} + \eta_\varepsilon u^\varepsilon_{x_2x_2} = 0 \quad \text{in } \Omega, \tag{8}$$

and $u^\varepsilon = g$ on Ω. By the maximum principle, we have $\sup_\Omega |u^\varepsilon| \leq \sup_{\partial\Omega} |g|$. We will establish proper uniform norms of u^ε and obtain the desired solution by sending $\varepsilon \to 0$.

Our proof of this part is different from [7] which uses Caffarelli-Gutiérrez's Harnack inequality [5] to obtain uniform interior Hölder norms of those approximating solutions. Instead, we establish an interior bound of $u^\varepsilon_{x_2}$ first, as in Daskalopoulos-Savin [10]. In view of the standard uniformly elliptic equation theory, we only need to concern about the area near $\{x_1 = 0\}$. Suppose that $0 \in \Omega$ and $B_\tau \subset \Omega$ for some small $\tau > 0$. We shall show that $\|u^\varepsilon_{x_2}\|_{L^\infty(B_{\tau/2})} \leq C$ for some C independent of ε.

We claim that there exists a large universal constant β such that

$$L_\varepsilon(\beta(u^\varepsilon)^2 + (\varphi u^\varepsilon_{x_2})^2) \geq 0 \quad \text{in } \Omega, \tag{9}$$

where φ is some cutoff function in B_τ satisfying $\varphi = 1$ in $B_{\tau/2}$, $\varphi = 0$ in $\Omega \setminus B_\tau$, and $\varphi_{x_1} = 0$ for all $|x_1| \leq \tau/4$.

Indeed, a simple computation yields

$$L_\varepsilon(u^\varepsilon)^2 = 2((u^\varepsilon_{x_1})^2 + \eta_\varepsilon(u^\varepsilon_{x_2})^2)$$

and

$$\begin{aligned} L_\varepsilon(\varphi u^\varepsilon_{x_2})^2 &= L_\varepsilon\varphi^2(u^\varepsilon_{x_2})^2 + \varphi^2 L_\varepsilon(u^\varepsilon_{x_2})^2 + 2(\varphi^2)_{x_1}((u^\varepsilon_{x_2})^2)_{x_1} + 2\eta_\varepsilon(\varphi^2)_{x_2}((u^\varepsilon_{x_2})^2)_{x_2} \\ &= L_\varepsilon\varphi^2(u^\varepsilon_{x_2})^2 + 2\varphi^2((u^\varepsilon_{x_2x_1})^2 + \eta_\varepsilon(u^\varepsilon_{x_2x_2})^2) + 8(\varphi_{x_1}u^\varepsilon_{x_2})(\varphi u^\varepsilon_{x_2x_1}) \\ &\quad + 8\eta_\varepsilon(\varphi_{x_2}u^\varepsilon_{x_2})(\varphi u^\varepsilon_{x_2x_2}). \end{aligned}$$

Hence,

$$\begin{aligned} L_\varepsilon(\beta(u^\varepsilon)^2 + (\varphi u^\varepsilon_{x_2})^2) \geq& 2\beta\eta_\varepsilon(u^\varepsilon_{x_2})^2 + 2\varphi^2((u^\varepsilon_{x_2x_1})^2 + \eta_\varepsilon(u^\varepsilon_{x_2x_2})^2) \\ &+ L_\varepsilon\varphi^2(u^\varepsilon_{x_2})^2 + 8(\varphi_{x_1}u^\varepsilon_{x_2})(\varphi u^\varepsilon_{x_2x_1}) + 8\eta_\varepsilon(\varphi_{x_2}u^\varepsilon_{x_2})(\varphi u^\varepsilon_{x_2x_2}). \end{aligned}$$

By the Cauchy inequality and the facts

$$L_\varepsilon(\varphi^2) \geq -C_1\eta_\varepsilon, \quad |\varphi_{x_1} u^\varepsilon_{x_2}| \leq C_1\eta_\varepsilon |u^\varepsilon_{x_2}|,$$

the claim follows for large β independent of ε.

By (9) and the maximum principle, we have

$$\sup_{B_{\tau/2}} |u^\varepsilon_{x_2}| \leq \beta^{1/2} \sup_{\Omega} |u^\varepsilon|.$$

Since $Lu^\varepsilon_{x_2} = 0$, the same arguments can be applied inductively to show that $\partial^k u^\varepsilon / \partial x_2^k$ are bounded in the interior of Ω for any $k \in \mathbb{Z}^+$. Since $|u^\varepsilon_{x_2x_2}| \leq C$ for some C independent of ε and $u^\varepsilon_{x_1x_1} + \eta_\varepsilon u^\varepsilon_{x_2x_2} = 0$, we have

$$|u^\varepsilon_{x_1}| \leq C \int_{-1}^{1} \eta_\varepsilon(x_1)\,\mathrm{d}x_1 + C,$$

where we used the fact that $u^\varepsilon_{x_1}$ is bounded uniformly for $B_{3\tau/4} \cap \{x||x_1| \geq \tau/4\}$. Since $\alpha > -1$, the integral $\int_{-1}^{1} \eta_\varepsilon(x_1)\,\mathrm{d}x_1$ can be bounded independent of ε. The same arguments would show that $u^\varepsilon_{x_1x_2}$ and $u^\varepsilon_{x_1x_2x_2}$ are bounded as well.

For $\alpha \in (-1, 0)$ and any point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in B_{\tau/4}$, by the Taylor's formula we have

$$\begin{aligned}
&u^\varepsilon(x_1, \bar{x}_2) \\
&= u^\varepsilon(\bar{x}_1, \bar{x}_2) + u^\varepsilon_{x_1}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1) + (x_1 - \bar{x}_1)^2 \int_0^1 (1-\lambda) u^\varepsilon_{x_1x_1}(\xi_\lambda, \bar{x}_2)\,\mathrm{d}\lambda \\
&= u^\varepsilon(\bar{x}_1, \bar{x}_2) + u^\varepsilon_{x_1}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1) - (x_1 - \bar{x}_1)^2 \int_0^1 (1-\lambda) u^\varepsilon_{x_2x_2}(\xi_\lambda, \bar{x}_2)\eta(\xi_\lambda)\,\mathrm{d}\lambda \\
&= u^\varepsilon(\bar{x}_1, \bar{x}_2) + u^\varepsilon_{x_1}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1) - u^\varepsilon_{x_2x_2}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1)^2 \int_0^1 (1-\lambda)\eta(\xi_\lambda)\,\mathrm{d}\lambda \\
&\quad + O(|x_1 - \bar{x}_1|^3 \int_0^1 \eta(\xi_\lambda)\,\mathrm{d}\lambda),
\end{aligned}$$

where $\xi_\lambda = \bar{x}_1 + \lambda(x_1 - \bar{x}_1)$. One should note that $\int_0^1 \eta(\xi_\lambda)\,\mathrm{d}\lambda \leq C|x_1 - \bar{x}_1|^\alpha$ for some constant $C > 0$ independent of ε. Making use of Taylor's formula again, we have

$$\begin{aligned}
u^\varepsilon(x_1, x_2) =& u^\varepsilon(x_1, \bar{x}_2) + u^\varepsilon_{x_2}(x_1, \bar{x}_2)(x_2 - \bar{x}_2) + \frac{1}{2} u^\varepsilon_{x_2x_2}(\bar{x}_1, \bar{x}_2)(x_2 - \bar{x}_2)^2 \\
&+ O(|x_2 - \bar{x}_2|^3 + |(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)^2|),
\end{aligned}$$

and

$$u^\varepsilon_{x_2}(x_1, \bar{x}_2) = u^\varepsilon_{x_2}(\bar{x}_1, \bar{x}_2) + u^\varepsilon_{x_1x_2}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1) + O(|x_1 - \bar{x}_1|^{2+\alpha}).$$

Therefore,

$$|u^\varepsilon(x_1,x_2) - u^\varepsilon(x_1,\bar{x}_2) - u^\varepsilon_{x_1}(\bar{x}_1,\bar{x}_2)(x_1-\bar{x}_1) - u^\varepsilon_{x_2}(\bar{x}_1,\bar{x}_2)(x_2-\bar{x}_2)| \le C|x-\bar{x}|^{2+\alpha}.$$

By the arbitrary choice of $\bar{x}$, we conclude that

$$\|u^\varepsilon\|_{C^{1,1+\alpha}(B_{\tau/4})} \le C. \tag{10}$$

The same argument is also applicable to $\alpha \ge 0$, and one can conclude that

$$\|u^\varepsilon\|_{C^{2,\delta}(B_{\tau/4})} \le C \tag{11}$$

for some $\delta > 0$ depending only on α.

By passing to a subsequence, we obtain a strong solution u of (5) and u satisfies (7). □

Remark 3.3. *From the proof of Proposition 3.2, we see that:*

- *If* $\alpha \in (-1,0)$, $u \in C^{1,1+\alpha}_{loc}(\Omega)$;
- *If* $\alpha \ge 0$, $u \in C^{2,\delta}_{loc}(\Omega)$ *for some* $\delta > 0$ *depending only on* α.

Let

$$\phi(x_1,x_2) = |x_1|^{2+\alpha} + x_2^2 \quad \text{in } \mathbb{R}^2. \tag{12}$$

Then

$$\nabla^2\phi = \begin{pmatrix} (2+\alpha)(1+\alpha)|x_1|^\alpha & 0 \\ 0 & 2 \end{pmatrix},$$

and

$$(\nabla^2\phi)^{1/2} = \begin{pmatrix} \sqrt{(2+\alpha)(1+\alpha)}|x_1|^{\alpha/2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Hence, $\det \nabla^2\phi = c(\alpha)|x_1|^\alpha$, where $c(\alpha) = 2(\alpha+2)(\alpha+1) > 0$. For any $x \in \mathbb{R}^2$ and $t > 0$, denote

$$S(x,t) = S_\phi(x,t) = \{y \in \mathbb{R}^2 | \phi(y) < \ell(y) + t\},$$

where $\ell(y)$ is the support plane of ϕ at $(x,\phi(x))$. It is direct to verify

Condition μ_∞ [5]: *For any given* $\delta_1 \in (0,1)$, *there exists* $\delta_2 \in (0,1)$ *such that, for all sections* S *and all small subsets* $E \subset S$,

$$\frac{|E|}{|S|} < \delta_2 \quad \textit{implies} \quad \frac{\int_E |x_1|^\alpha \,\mathrm{d}x}{\int_S |x_1|^\alpha \,\mathrm{d}x} < \delta_1. \tag{13}$$

Let

$$A(x_1, x_2) = \begin{pmatrix} |x_1|^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly,

$$B := (\nabla^2 \phi)^{1/2} A (\nabla^2 \phi)^{1/2} = \begin{pmatrix} (2+\alpha)(1+\alpha) & 0 \\ 0 & 2 \end{pmatrix},$$

which is positive definite if $\alpha > -1$. Therefore, we can apply Caffarelli-Gutiérrez's Harnack inequality [5] to obtain the following proposition.

Proposition 3.4. *Let $u \geq 0$ be a strong solution of*

$$Lu = 0 \quad \textit{in } S(x_0, 2),$$

where x_0 is an arbitrary point in $\mathbb{R}^2$. Then there exists a positive constant β depending only on α such that

$$\sup_{S(x_0,1)} u \leq \beta \inf_{S(x_0,1)} u.$$

Corollary 3.5. *Let u be a strong solution of*

$$Lu = 0 \quad \textit{in } S(0, 2).$$

Then there exist constants $C > 0$ and $\gamma \in (0, 1)$ depending only on α such that

$$\|u\|_{C^\gamma(S(0,1))} \leq C \|u\|_{L^\infty(S(0,2))}.$$

Theorem 3.6. *Let u be a nonnegative strong solution of*

$$Lu = 0 \quad \textit{in } \mathbb{R}^2. \tag{14}$$

Then u is a constant in $\mathbb{R}^2$.

Proof. Consider the scaling $u_r = \frac{1}{r} u(r^{1/(2+\alpha)} x_1, r^{1/2} x_2)$ for $r > 0$. Then u_r also satisfies (14). By Proposition 3.4, we have

$$\sup_{S(0,2)} u_r \leq \beta u_r(0).$$

It follows from Corollary 3.5 that

$$[u_r]_{C^\gamma(S(0,1))} \leq C\beta u_r(0).$$

For any two distinct points x, y in $\mathbb{R}^2$, we have, for sufficiently large r,

$$\begin{aligned} |u(x) - u(y)| &= r|u_r(r^{-1/(2+\alpha)} x_1, r^{-1/2} x_2) - u_r(r^{-1/(2+\alpha)} y_1, r^{-1/2} y_2)| \\ &\leq r[u_r]_{C^\gamma(S(0,1))} |r^{-2/(2+\alpha)} (x_1 - y_1)^2 + r^{-1}(x_2 - y_2)^2|^{\gamma/2} \\ &\leq C\beta u(0) |r^{-2/(2+\alpha)} (x_1 - y_1)^2 + r^{-1}(x_2 - y_2)^2|^{\gamma/2}. \end{aligned}$$

Sending $r \to \infty$, we obtain $u(x) = u(y)$. The proof is completed. $\square$

4 Regularity for solutions of degenerate Monge-Ampère equations

Define the measure μ_α in $\mathbb{R}^2$ as $\mathrm{d}\mu_\alpha = |x_1|^\alpha \mathrm{d}x_1\mathrm{d}x_2$ for $\alpha > -1$. For any bounded open convex set $\Omega \subset \mathbb{R}^2$, it is clear that the measure μ_α has the *doubling property* in Ω, i.e., there exists a constant $c_\alpha > 0$, depending only on α and Ω, such that for any $(\bar{x}_1, \bar{x}_2) \in \Omega$ and any ellipsoids $E \subset \mathbb{R}^2$ centered at origin with $(\bar{x}_1, \bar{x}_2) + E \in \Omega$ there holds

$$\mu_\alpha((\bar{x}_1, \bar{x}_2) + E) \geq c_\alpha \mu_\alpha(((\bar{x}_1, \bar{x}_2) + 2E) \cap \Omega). \tag{15}$$

Consequently, we have the following theorem.

Theorem 4.1. *Let Ω be an open convex set in $\mathbb{R}^2$, and u be the generalized solution of*

$$\det \nabla^2 u(x) = |x_1|^\alpha \quad \textit{in } \Omega,$$

with $u = 0$ on $\partial\Omega$. Then u is strictly convex in Ω, $u \in C^{1,\delta}_{loc}(\Omega)$ for some $\delta > 0$ depending only on α. Furthermore, the partial Legendre transform u^* of u is a strong solution of

$$Lu^* = 0 \quad \textit{in } T(\Omega),$$

where the map T is given in (4)*.*

Proof. The strict convexity and the $C^{1,\delta}$ regularity was proved in [2, 3]. Hence, T is continuous and one-to-one, and thus, $T(\Omega)$ is open. Let $u_k \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ be the solution of

$$\det \nabla^2 u_k = \eta_{1/k}(x_1) \quad \text{in } \Omega \tag{16}$$

with $u_k = 0$ on $\partial\Omega$, where $\eta_{1/k}(x_1)$ is the same as the one in the proof of Proposition 3.2 with $\varepsilon = 1/k$. Let

$$T_k : \Omega \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto (x_1, \partial_2 u_k(x)),$$

and u_k^* be the partial Legendre transform of u_k. Then u_k^* satisfies (8). Clearly, up to a subsequence, $u_k \to u$ in $C^1_{loc}(\Omega)$ as $k \to \infty$. Thus, $\lim_{k\to\infty} T_k(x) = T(x)$ for any $x \in \Omega$, and for any $y \in T(\Omega)$ there exists λ sufficiently small such that $B_\lambda(y) \subset T(\Omega) \cap T_k(\Omega)$ for every large k. By the same argument used in proof of Proposition 3.2, we can conclude that $u^* \in C^1(T(\Omega)) \cap C^2(T(\Omega) \setminus \{x_1 = 0\})$ and satisfies $Lu^* = 0$ in $T(\Omega) \setminus \{x_1 = 0\}$. □

Theorem 4.2. *Let u be a generalized solution of* (2)*. Then u is strictly convex.*

Proof. By the two dimensional Monge-Ampère equation theory, if u is a generalized solution of

$$\det \nabla^2 u \geq c_0 > 0 \quad \text{in } \Omega,$$

where Ω is an open set in $\mathbb{R}^2$, then u is locally strictly convex in Ω. Hence, we only need to consider the situation $\alpha > 0$. After subtracting a supporting plane of u at origin, we may assume that

$$u \geq 0 \quad \text{in } \mathbb{R}^2 \quad \text{and } u(0) = 0.$$

Claim: There exists a sufficiently large $R > 0$ such that

$$\min_{\partial B_R} u > 0. \tag{17}$$

Indeed, if not, namely, $\min_{\partial B_R} u = 0$ for all sufficiently large $R > 0$. The strict convexity of u away from $\{x_1 = 0\}$ implies $u(Re_2) = 0$ or $u(-Re_2) = 0$, where $e_2 = (0, 1)$. Without loss of generality, we may assume $u(Re_2) = 0$. Let

$$M = \max_{\partial B_1} u > 0,$$

and Δ be the triangle generated by the segment $\{(x_1, 0)||x_1| \leq 1\}$ and the point Re_2. By the convexity of u, we have

$$M \geq u \quad \text{in } \Delta.$$

It is clear that the ellipsoid

$$E = \{(x_1, x_2) : x_1^2 + \frac{1}{R^2}(x_2 - R/4)^2 = \frac{1}{16}\}$$

sits in Δ. Let

$$u_R(y_1, y_2) = \frac{1}{R} u(y_1, R(y_2 + 1/4)).$$

We have

$$\det \nabla^2 u_R(y_1, y_2) = |y_1|^\alpha \quad \text{in } B_{1/4},$$

and $u_R \leq \frac{M}{R}$ in $B_{1/4}$. Choosing a small constant $\tau > 0$, depending only on α, such that

$$S_\phi(0, \tau) \subset B_{1/4},$$

where ϕ is given in (12). By the comparison principle (see, e.g., [15]),

$$0 \leq u_R \leq \sqrt{c(\alpha)^{-1}}(\phi - \tau) + \max_{\partial S(0,\tau)} u_R \quad \text{in } S_\phi(0, \tau),$$

where $c(\alpha) = 2(\alpha + 2)(\alpha + 1)$. In particular,

$$0 \leq -\sqrt{c(\alpha)^{-1}}\tau + \max_{\partial S(0,\tau)} u_R \leq -\sqrt{c(\alpha)^{-1}}\tau + M/R.$$

That is

$$R \leq \frac{\sqrt{c(\alpha)}M}{\tau},$$

which contradicts to the assumption that R can be arbitrarily large.

Thus, (17) holds and we can conclude Theorem 4.2 from Theorem 4.1. $\square$

One might ask if every solution of

$$\det \nabla^2 u = |x_1|^\alpha \quad \text{in } B_1 \subset \mathbb{R}^2$$

is strictly convex, where $\alpha > 0$. The following example shows that this is not the case.

Example 4.3. *It is clear that for every $\alpha > 0$ there always exists a positive convex smooth solution w of the ODE*

$$\begin{cases} \frac{\alpha(\alpha+2)}{4} w(t)w(t)'' - \frac{(\alpha+2)^2}{4}(w'(t))^2 = 1, \\ w(0) = 1, \\ w'(0) = 1, \end{cases} \tag{18}$$

near $t = 0$. Then $u = |x_1|^{\frac{\alpha+2}{2}} w(x_2)$ is a generalized solutions of $\det \nabla^2 u = |x_1|^\alpha$ *in a small open set in $\mathbb{R}^2$. But u is not strictly convex (is smooth for certain α, though). By proper scaling and translation we can make the equation holds in B_1.*

Proof of Theorem 1.1. Let u be a generalized solution of (2). It follows from Theorem 4.2 that u is strictly convex, and hence u is smooth away from $\{x_1 = 0\}$. By Theorem 4.1, we know that $u \in C^{1,\delta}_{loc}(\mathbb{R}^2)$ and the partial Legendre transform u^* of u is a strong solution of

$$Lu^* = u^*_{11} + |p_1|^\alpha u^*_{22} = 0 \quad \text{in } T(\mathbb{R}^2), \tag{19}$$

where $u^*_{ii} = u^*_{p_i p_i}$ and $T(x_1, x_2) = (x_1, u_{x_2}(x_1, x_2)) = (p_1, p_2)$. Moreover, T is continuous and one-to-one.

Given Theorem 3.6 and Proposition 3.4, the rest of the proof is similar to that in Section 2 for $\alpha = 0$.

Step 1: Prove the theorem under the assumption: $T(\mathbb{R}^2) = \mathbb{R}^2$.

Since u^* is convex with respect to p_2, we have that $u^*_{22} \geq 0$. Note that $Lu^*_{22} = 0$ in $\mathbb{R}^2$. By Theorem 3.6, $u^*_{22} \equiv a$ for some nonnegative constant a. By the equation $Lu^* = 0$, we have $u^*_{11} = -a|p_1|^\alpha$. Hence, $u^*_{121} \equiv u^*_{122} \equiv 0$ in $\{p_1 > 0\}$. Consequently, $u^*_{12} \equiv b$ in $\{p_1 > 0\}$ for some constant b. It follows from calculus that

$$u^* = -\frac{a}{(\alpha+1)(\alpha+2)}|p_1|^{2+\alpha} + \frac{a}{2}p_2^2 + bp_1p_2 + \ell(p_1, p_2) \tag{20}$$

for some linear function ℓ in $\{p_1 > 0\}$. The same argument applies to $\{p_1 < 0\}$. Since $u^*, u^*_2 \in C^1(\mathbb{R}^2)$, (20) holds for all $p \in \mathbb{R}^2$. Since $u = (u^*)^*$, $a > 0$ and we are done.

Step 2: Prove: $T(\mathbb{R}^2) = \mathbb{R}^2$.

We prove it by contradiction. Suppose that there exists $\bar{x}_1$ such that $\lim_{x_2 \to \infty} u_2(\bar{x}_1, x_2) := \beta_2 < \infty$. Then, as in Section 2, $\lim_{x_2 \to \infty} u_2(x_1, x_2) = \beta$ for every $x_1 \in \mathbb{R}$, and we may assume

$\beta = 1$. Therefore, $T(\mathbb{R}^2) = (-\infty, \infty) \times (\beta_0, 1)$ for some $-\infty \leq \beta_0 < 1$. Since T is one-to-one and $u_2^*(p_1, p_2) = x_2$, we have $\lim_{p_2 \to 1-} u_2^*(p_1, p_2) = \infty$. The same argument in Section 2 shows that

$$\lim_{(p_1,p_2)\to(\bar{p}_1,1)} u_2^*(p_1, p_2) = +\infty$$

for any $\bar{p}_1 \in \mathbb{R}$.

Case 1: $\alpha \geq 0$.

Without loss of generality, we may assume that u_2^* is positive in $[1, 3] \times [0, 1)$. For any $C > 0$ large, we let

$$v(p_1, p_2) := u_2^*(p_1, p_2) - Cp_2(p_1 - 1)(3 - p_1) - \frac{C}{3}p_2^3 + \frac{C}{3}.$$

It is direct to check that $Lv < 0$ in $[1, 3] \times [0, 1)$. By the maximum principle, $v \geq 0$ in $[1, 3] \times [0, 1)$. In particular, $v(2, \bar{p}_2) \geq 0$ where $\bar{p}_2 \in (0, 1)$ is chosen such that

$$\bar{p}_2 + \bar{p}_2^3/3 - 1/3 = 1/2.$$

Hence, $u_2^*(2, \bar{p}_2) \geq C/2$ for all $C > 0$, which is a contradiction.

Case 2: $\alpha \in (-1, 0)$.

Without loss of generality, we may assume that u_2^* is positive in $[1/2, 1] \times [0, 1)$. For any $C > 0$ large, we let

$$v(p_1, p_2) := u_2^*(p_1, p_2) - Cp_2(p_1 - 1/2)(1 - p_1) - \frac{C}{3}p_2^3 + \frac{C}{3}.$$

It is direct to check that $Lv < 0$ in $[1, 3] \times [0, 1)$. By the maximum principle, $v \geq 0$ in $[1/2, 1] \times [0, 1)$. In particular, $v(3/4, \bar{p}_2) \geq 0$ where $\bar{p}_2 \in (0, 1)$ is chosen such that

$$\bar{p}_2/16 + \bar{p}_2^3/3 - 1/3 = 1/32.$$

Hence, $u_2^*(3/4, \bar{p}_2) \geq C/32$ for all $C > 0$, which is a contradiction.

The proof is completed. □

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